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Unit I - FOURIER SERIES

1. State Dirichlet's conditions for a given function to expand in Fourier series.

i) $f(x)$ is well defined, periodic and single valued.

ii) $f(x)$ has finite number of finite discontinuities and no infinite discontinuities.

iii) $f(x)$ has finite number of finite maxima and minima.

2. Write the formulas of Fourier constants for $f(x)$ in $(c, c+2l)$..

Solution:

$$a_0 = \frac{1}{l}\int_c^{c+2l} f(x)\,dx$$

$$a_n = \frac{1}{l}\int_c^{c+2l} f(x)\cos\frac{n\pi x}{l}\,dx$$

$$b_n = \frac{1}{l}\int_c^{c+2l} f(x)\sin\frac{n\pi x}{l}\,dx$$

3. Find the constant a_0 of the Fourier series for the function $f(x) = k,\quad 0 < x < 2\pi$.

Solution:

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{2\pi} k\,dx$$

$$= \frac{k}{\pi}[x]_0^{2\pi} = \frac{2\pi k}{\pi}$$

$$a_0 = 2k\,.$$

4. Given $f(x) = x^2,\ 0 \le x \le 2$ which one of the following is correct.

(a) an even function (b) an odd function (c) neither even nor odd

Ans: (a)

5. Find b_n in expanding $f(x) = x(2l - x)$ as Fourier series in the interval $(0,2l)$.

Solution:

$$b_n = \frac{1}{l}\int_0^{2l} x(2l-x)\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{1}{l}\left[(2lx - x^2)\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) - (2l-2x)\left(\frac{-\sin\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right) + (-2)\left(\frac{\cos\frac{n\pi x}{l}}{\frac{n^3\pi^3}{l^3}}\right)\right]_0^{2l}$$

$=0.$

6. In the expansion of $f(x) = \sinh x \quad (-\pi, \pi)$ **as a Fourier series find the coefficient of** a_n **.**

Solution: Given $f(x) = \sinh x$ **.** $\Rightarrow f(-x) = \sinh(-x) = -\sinh x = -f(x)$

$\therefore f(x)$ is an odd function. Hence $a_n = 0$.

7. **Find a_n in expanding** $f(x) = (1-x)^2$ **as Fourier series in the interval** $(0,1)$.

Solution:

$$a_n = 2\int_0^1 (1-x)^2 \cos n\pi x \, dx$$

$$= 2\left[(1-x)^2\left(\frac{\sin n\pi x}{n\pi}\right) - \left(2(1-x)(-1)\left(-\frac{\cos n\pi x}{n^2\pi^2}\right)\right) + 2\left(-\frac{\sin n\pi x}{n^3\pi^3}\right)\right]_0^1$$

$$= 2\left[(0-0+0) - \left(0 + 2\left(\frac{-1}{n^2\pi^2}\right)\right)\right] = \frac{4}{n^2\pi^2}.$$

8. Explain periodic function with example. Solution:

A function $f(x)$ is said to be periodic if there exists a number $T > 0$ such that $f(x+T) = f(x)$ for all x in the domain of the definition of function. The least value of T satisfying the above condition is called the fundamental period or simply period of the function $f(x)$

E.g. $f(x) = \sin x$

$f(x + 2\pi) = \sin(x + 2\pi) = \sin x = f(x)$

Hence $\sin x$ is a periodic function with period 2π.

9. In the expansion of $f(x) = \sin x \ (-\pi, \pi)$ **as a Fourier series find the coefficient of** a_n **.**

Solution: Given $f(x) = \sin x$

$f(-x) = \sin(-x) = -\sin x = -f(x)$

$\therefore f(x)$ is an odd function.Hence $a_n = 0$.

10. Find the Fourier constant a_n **for** $x \cos x$ **in (** $-\pi, \pi$ **).**

Solution: Let $f(x) = x \cos x$

$\therefore f(-x) = -x\cos(-x) = -x\cos x = -f(x)$

$\therefore f(x)$ is an odd function.

Hence $a_n = 0$.

11. Find the constant term a_0 and the coefficient a_n of $\cos nx$ in the Fourier series expansion of $f(x) = x - x^3$ in $(-\pi, \pi)$.

Solution: $f(x) = x - x^3$

$\Rightarrow f(-x) = -x + x^3 = -(x - x^3)$

$\therefore f(-x) = -f(x)$ is an odd function.

$\therefore \quad a_0 = 0 \;\; and \;\; a_n = 0$.

12. Find $a_{0,}\ a_n$ in expanding $f(x) = \sin ax$ as a Fourier series in $(-\pi, \pi)$.

Solution: $f(-x) = \sin a(-x) = \sin(-ax) = -\sin ax = -f(x)$

Hence $f(x)$ is an odd function.

$\therefore a_0 = 0$ and $a_n = 0$.

13. If $f(x) = |x|$ expanded as a Fourier series in $-\pi < x < \pi$ .Find a_0 .

Solution:

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{\pi}\int_{-\pi}^{\pi} |x|dx = \frac{2}{\pi}\int_0^{\pi} xdx \quad [\; |x| \text{ is an even function}]$$

$$= \frac{2}{\pi}\left[\frac{x^2}{2}\right]_0^{\pi} = \pi \qquad \Rightarrow a_0 = \pi$$

14. Find the constants b_n for $f(x) = |x|$ in $-\pi < x < \pi$.

Solution: Given $f(x) = |x|$

$f(-x) = |-x| = |x|$

$\therefore f(x) = f(-x)$

$\therefore f(x)$ is an even function. Hence b_n =0.

15. Find b_n in the expansion of x^2 as a Fourier series in ($-\pi, \pi$) .

Solution:

Given $f(x) = x^2$

Now $f(-x) = (-x)^2 = x^2 = f(x)$

$\therefore f(x)$ is an even function. Hence b_n =0.

16. Find the Fourier constant b_n for $x \sin x$ in ($-\pi, \pi$) .

Solution: Let $f(x) = x \sin x$

Therefore $f(x)$ is even function of x in $(-\pi, \pi)$

The Fourier series of $f(x)$ contains cosine terms only.

Which implies b_n=0.

17. Does $f(x) = \tan x$ possess a Fourier expansion in $(0, \pi)$.

Solution: $f(x) = \tan x$ has an infinite discontinuity at $x = \frac{\pi}{2}$.

Since the Dirichlet's conditions on continuity is not satisfied, the Function $f(x) = \tan x$ has no Fourier expansion.

18. If $f(x) = x^2 + x$ is expressed as a Fourier series in the interval (-2,2), to which value this series converges at x =2.

Solution:

The value of the Fourier series of $f(x)$ at x =2 is

$$\frac{1}{2}[f(-2) + f(2)] = \frac{1}{2}[4 - 2 + 4 + 2] = 4$$

19. If f(x) is an odd function defined in (-l,l) , what are the values of a_0 and a_n ? Solution:

Since f(x) is an odd function of x in (-l,l) , its Fourier expansion contains sine terms only. $\therefore a_5 = 0$ and a_n=0.

20. If f(x) is discontinuous at x =a, what does its Fourier series represent at the point? [or] Define the value of the fourier series of $f(x)$ at a point of discontinuity.

Solution:

The value of the Fourier series at x=a is

$$f(a) = \frac{1}{2}[\lim_{h\to 0} f(a+h) + \lim_{h\to 0} f(a-h)]$$

$$= \frac{1}{2}[f(a-) + f(a+)].$$

21. If $f(x) = \begin{cases} \cos x & ,0 < x < \pi \\ 50 & ,\pi < x < 2\pi \end{cases}$ and f(x+2 π) $= f(x)$ for all x, find the sum of the Fourier series of f(x) at x= π?

Solution:

$$f(\pi) = \frac{1}{2}[f(\pi-) + f(\pi+)] = \frac{1}{2}[\cos\pi + 50] = \frac{49}{2}.$$

22. If the Fourier series for the function f(x)= $\begin{cases} 0, & 0 < x < \pi \\ \sin x, & \pi < x < 2\pi \end{cases}$ **is**

f(x)= $\frac{\sin x}{2} - \frac{1}{\pi} + \frac{2}{\pi}\left[\frac{\cos 2x}{1.3} + \frac{\cos 4x}{3.5} + \frac{\cos 6x}{5.7} + \ldots\ldots\right]$ **deduce that**

$\frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} + \ldots\ldots\ldots\infty = \frac{\pi - 2}{4}$.

Solution:

Put x = $\frac{\pi}{2}$ in the Fourier expansion of f(x),

$$f\left(\frac{\pi}{2}\right) = \frac{1}{2} - \frac{1}{\pi} + \frac{2}{\pi}\left[-\frac{1}{1.3} + \frac{1}{3.5} - \frac{1}{5.7} + \ldots\right]$$

$$\Rightarrow \frac{2}{\pi}\left[\frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} + \ldots\right] = \frac{1}{2} - \frac{1}{\pi} \quad \text{Since } f\left(\frac{\pi}{2}\right) = 0$$

$$\therefore \frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} + \ldots\ldots = \frac{\pi - 2}{2\pi} \times \frac{\pi}{2} = \frac{\pi - 2}{4}.$$

23. Suppose the function $x \cos x$ **has the series expansion** $\sum_{1}^{\infty} b_n \sin nx$ **in** $(\pi, -\pi)$**, find the value of** b_1 **.**

Solution: $b_1 = \frac{2}{\pi}\int_0^{\pi} x \cos x \sin x dx$

$$= \frac{1}{\pi}\int_0^{\pi} x \sin 2x dx \qquad = \frac{1}{\pi}\left[x\left(\frac{-\cos 2x}{2}\right) + \frac{\sin 2x}{4}\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[-\frac{\pi}{2}\right] = -\frac{1}{2}.$$

24. Find the constant term in the Fourier expansion of $f(x) = \cos^2 x$ **in** $(-\pi, \pi)$ **.**

Solution:

$\cos^2 x$ is an even function of x in $(-\pi, \pi)$.

$$\therefore \cos^2 x = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

$$\frac{1 + \cos 2x}{2} = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \;\therefore \text{The constant term is } a_0 = 1$$

25. Find the Fourier constants b_n for $x \sin x$ in $(-\pi, \pi)$.

Solution:

Given $f(x) = x \sin x$

Now $f(-x) = -x \sin(-x) = x \sin x = f(x)$

$\therefore$ $f(x)$ is an even function. $\therefore$ $b_n = 0$.

26. Find a_n, in expanding e^{-x} as Fourier series in $(-\pi, \pi)$.

Solution:

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} e^{-x}\cos nx\,dx = \frac{1}{\pi}\left[\frac{e^{-x}}{1+n^2}\left(-\cos nx + n\sin nx\right)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{e^{-\pi}}{1+n^2}\left(-(-1)^n\right) - \frac{e^{\pi}}{1+n^2}\left(-(-1)^n\right)\right] = \frac{(-1)^n}{\pi(1+n^2)}\left[e^{\pi} - e^{-\pi}\right]$$

27. Define root mean square value of a function $f(x)$ in $a < x < b$.

Solution:

$$\text{R.M.S value } \bar{y} = \sqrt{\frac{1}{b-a}\int_a^b \left[f(x)\right]^2 dx.}$$

28. Find the root mean square value of the function $f(x) = x$ in the interval (0, l).

Solution:

$$\text{R.M.S value} \quad \bar{y} = \sqrt{\frac{1}{l}\int_0^l x^2\,dx} = \sqrt{\frac{1}{l}\left[\frac{l^3}{3}\right]} = \frac{l}{\sqrt{3}}$$

29. Find the R.M.S value of the function $f(x) = x$ in $(0, \pi)$.

Solution:

$$\text{R.M.S} = \sqrt{\frac{\int_0^{\pi} x^2\,dx}{\pi}} = \sqrt{\frac{\left(\frac{x^3}{3}\right)_0^{\pi}}{\pi}} = \sqrt{\frac{\pi^3}{3\pi}} = \frac{\pi}{\sqrt{3}}.$$

30. Find the root mean square value of the function $f(x) = x^2$ in the interval (-1,1)

Solution:

$$\bar{y} = \sqrt{\frac{1}{2l}\int_{-l}^{l} x^4\,dx} = \sqrt{\frac{1}{2}\left[\frac{x^5}{5}\right]_{-1}^{1}} = \sqrt{\frac{1}{2}\left[\frac{2}{5}\right]} = \frac{1}{\sqrt{5}}.$$

31. Find the R.M.S value of the function $f(x) = x^2 \quad in \quad (0,\pi)$.

Sol: $\bar{y} = \sqrt{\frac{1}{\pi}\int_0^{\pi} x^4\, dx} = \sqrt{\frac{1}{\pi}\left[\frac{x^5}{5}\right]_0^{\pi}} = \frac{\pi^2}{\sqrt{5}}$

32. State parseval's identity (Theorem) of Fourier series.

Solution:

If f(x) has a Fourier series of the form

$$f(x) = \frac{a_0}{2} + \sum_1^{\infty} (a_n \cos nx + b_n \sin nx) \text{ in } (0,2\pi), \text{ then}$$

$$\frac{1}{2\pi}\int_0^{2\pi} [f(x)]^2\, dx = \frac{a_0^2}{4} + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2 + b_n^2).$$

33. State parseval's identity for the half –range cosine expansion of f(x) in ($0, l$ **).**

Solution:

$$\frac{2}{l}\int_0^{l}[f(x)]^2\, dx = \frac{a_0^2}{2} + \sum_1^{\infty} a_n^2.$$

34. If the Fourier series of the function f(x) =x+x² in the interval ($-\pi,\pi$ **) is** $\frac{\pi^2}{3} + \sum_1^{\infty}(-1)^n\left(\frac{4}{n^2}\cos nx - \frac{2}{n}\sin nx\right)$, **then find the value of the infinite series** $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots\ldots$

Solution: Given $f(x) = \frac{\pi^2}{3} + \sum_1^{\infty}(-1)^n\left(\frac{4}{n^2}\cos nx - \frac{2}{n}\sin nx\right)$

Put $x = \pi$, $f(\pi) = \frac{\pi^2}{3} + \sum_1^{\infty}\frac{4}{n^2}$

$$f(\pi) = \frac{1}{2}[f(-\pi) + f(\pi)] = \frac{1}{2}[-\pi + \pi^2 + \pi + \pi^2] = \pi^2$$

$$\sum_1^{\infty}\frac{}{n} \qquad \sum_1^{\infty}\frac{}{n} \qquad \frac{= 2\pi^2}{3}$$

$$\therefore \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots\ldots = \frac{2\pi^2}{3\times 4} = \frac{\pi^2}{6}$$

35. State Parseval's identity for full range expansion of $f(x)$ as Fourier series in $(0,2l)$.

Solution:

Let f(x) be a periodic function with period 2l defined in the interval $(0,2l)$.

then $$\frac{1}{2l}\int_0^{2l}\left[f(x)\right]^2 dx = \frac{a_0^2}{4} + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2 + b_n^2).$$

36. If the Fourier series corresponding to *f(x) = x* in the interval $(0,2\pi)$ is $\frac{a_0}{2} + \sum_1^{\infty}(a_n \cos nx + b_n \sin nx)$ without finding the value of a_0, a_n, b_n. Find the values of

$$\frac{a_0^2}{2} + \sum_1^{\infty}(a_n^2 + b_n^2)$$

Solution: By parseval's identity

$$\frac{a_0^2}{2} + \sum_1^{\infty}(a_n^2 + b_n^2) = 2.\frac{1}{2\pi}\int_0^{2\pi} x^2\, dx = \frac{1}{\pi}\left[\frac{x^3}{3}\right]_0^{2\pi} = \frac{8\pi^2}{3}.$$

37. If $\cos^3 t = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nt + b_n \sin nt)$ **in** $0 \le t \le 2\pi$**, find the sum of the series**

$$\frac{a_0^2}{4} + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2 + b_n^2).$$

Solution: $$\frac{a_0^2}{4} + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2 + b_n^2) = \frac{1}{2\pi}\int_0^{2\pi}\cos^6 t\,dt.$$

$$= \frac{4}{2\pi}\int_0^{\pi/2}\cos^6 t\,dt$$

$$= \frac{2}{\pi}\left[\frac{5}{6}.\frac{3}{4}\frac{1}{2}\frac{\pi}{2}\right] = \frac{5}{16}.$$

38. Find the value of a_n in the cosine series expansion of f(x)=k in the interval(0,10). Solution:

Here l =10 and f(x) =k

$$a_n = \frac{2}{10}\int_0^{10} k\cos\frac{n\pi x}{10}\,dx$$

$$= \frac{k}{5}\left[\frac{\sin\frac{n\pi x}{10}}{\frac{n\pi}{10}}\right]_0^{10} = \frac{2k}{n\pi}[\sin n\pi - \sin 0] = 0 .$$

39. Find the coefficient a_5 of $\cos 5x$ in the Fourier cosine series of the function f(x) = $\sin 5x$ in the interval (0, 2π).

Solution:

$$a_5 = \frac{1}{\pi}\int_0^{2\pi}\sin 5x\cos 5x\,dx = \frac{1}{2\pi}\int_0^{2\pi}\sin 10x\,dx$$

$$a_5 = \frac{1}{2\pi}\left[\frac{-\cos 10x}{10}\right]_0^{2\pi} = 0 .$$

40. In the Fourier expansion of

$$f(x) = \begin{cases} 1+\frac{2x}{\pi}, & -\pi < x < 0 \\ 1-\frac{2x}{\pi}, & 0 < x < \pi \end{cases}$$

in $(-\pi,\pi)$. **find the value of** b_n, **the coefficient of** $\sin nx$.

Solution:

$$f(x) = \begin{cases} 1+\frac{2x}{\pi}, & -\pi < x < 0 \\ 1-\frac{2x}{\pi}, & 0 < x < \pi \end{cases}$$

$$\therefore f(x) = \begin{cases} 1-\frac{2x}{\pi}, & 0 < x < \pi \\ 1+\frac{2x}{\pi}, & -\pi < x < 0 \end{cases}$$

$$\Rightarrow f(-x) = f(x)$$

$\therefore f(x)$ Is an even function in $(-\pi,\pi)$.

Hence the coefficient $b_n = 0$, since the Fourier series contains only cosine terms.

41. Find the value of a_n in the cosine series expansion of f(x)=5 in the interval(0,8). Solution:

Here l =8 and f(x) =5

$$a_n = \frac{2}{4}\int_0^8 5\cos\frac{n\pi x}{4}dx$$

$$= \frac{5}{4}\left[\frac{\sin\frac{n\pi x}{4}}{\frac{n\pi}{4}}\right]_0^8 = \frac{5}{n\pi}\left[\sin 2n\pi - \sin 0\right] = 0 .$$

42. Find the sine series for the function f(x) =1, 0<x< π .
Solution:

Given f(x) =1

$$b_n = \frac{2}{\pi}\int_0^{\pi}\sin nx\,dx$$

$$= \frac{2}{\pi}\left[-\frac{\cos nx}{n}\right]_0^{\pi} = \frac{2}{n\pi}[1-(-1)$$

$\therefore$ The Fourier sine series of f(x) = $\sum_1^{\infty} b_n \sin nx.$

$$f(x) = \frac{4}{\pi}[\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + ...].$$

43. Find the sine series for $f(x) = \begin{cases} 1, & 0 \le x \le \frac{l}{2} \\ 0, & \frac{l}{2} \le x \le l \end{cases}$

Solution: $f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$ dx

$$b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}dx = \frac{2}{l}\left[\int_0^{\frac{l}{2}}\sin\frac{n\pi x}{l}dx + \int_{\frac{l}{2}}^{l}(0)dx\right]$$

$$= \frac{2}{l}\left[\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right]_0^{\frac{l}{2}} = \frac{-2}{n\pi}\left[\cos\frac{n\pi}{2} - 1\right] = \frac{2}{n\pi}\left[1 - \cos\frac{n\pi}{2}\right]$$

$$\therefore f(x) = \sum_{n=1}^{\infty} \frac{2}{n\pi}\left[1 - \cos\frac{n\pi}{2}\right]\sin\frac{n\pi x}{l}$$

$$= \sum_{n=1}^{\infty} \frac{4}{n\pi}\sin^2\frac{n\pi}{4}\sin\frac{n\pi x}{l} \quad \left[1 - \cos\theta = 2\sin^2\frac{\theta}{2}\right]$$

44. Define Harmonic Analysis.

Ans: The process of finding the fourier series for a function given by numerical value is known as Harmonic Analysis.

45. If $f(x) = 2x \quad in \quad (0,4)$ then find the value of a_2 in the fourier series expansion.

Sol:

Given $f(x) = 2x \quad in \quad (0,4)$

$$a_n = \frac{1}{l}\int_0^{2l} f(x)\cos\frac{n\pi x}{l}\,dx$$

$$\therefore a_2 = \frac{1}{2}\int_0^4 2x\cos\frac{2\pi x}{2}\,dx$$

$$= \left[\frac{x\sin\pi x}{\pi} + \frac{\cos\pi x}{(\pi)^2}\right]_0^4 = \frac{1}{\pi^2} - \frac{1}{\pi^2} = 0\ .$$

Unit II -Fourier Transforms

Two Mark Question & Answers

1) State Fourier Integral Theorem

Sol:

If $f(x)$ is piecewise continuous, has piecewise continuous derivatives in every finite interval in $(-\infty, \infty)$ and absolutely integrable in $(-\infty, \infty)$,then

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t)e^{is(x-t)}\,dt\,ds \text{ (or) equivalently}$$

$$f(x) = \frac{1}{\pi}\int_{0}^{\infty}\int_{-\infty}^{\infty} f(t)\cos\{s(x-t)\}\,dt\,ds.$$

This is known as Fourier Integral of $f(x)$.

2) Define Fourier transform and its inverse transform. (or)Write the Fourier transform pair.

Sol: The Fourier transform of a function $f(x)$ is $F[f(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{isx}\,dx.$

The function $f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F[f(x)]e^{-isx}\,ds$ is called the inverse formula for the Fourier transform of $F[f(x)]$.

3) Define Fourier sine transform and its inverse.

Sol: Fourier sine transform of $f(x)$ is defined as $F_s[f(x)] = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\sin sx\,dx$.

Its inverse is defined by $f(x) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} F_s[f(x)]\sin sx\,ds.$

4) **Find the Fourier Transform of** $f(x) = \begin{cases} 1 & |x| \le 1 \\ 0 & |x| > 1 \end{cases}$

Sol:

We know that

$$F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx}\, dx.$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} e^{isx}\, dx = \frac{1}{\sqrt{2\pi}} \left[\frac{e^{isx}}{is} \right]_{-1}^{1}$$

$$= \frac{2}{\sqrt{2\pi}} \left[\frac{e^{is} - e^{-is}}{2is} \right] = \frac{2}{\sqrt{2\pi}} \frac{1}{2is} 2i \sin s = \frac{\sqrt{2}}{\pi} \frac{\sin s}{s}.$$

5) **Define Fourier cosine transform and its inverse.**

Sol: Fourier cosine transform of $f(x)$ is defined as $F_c[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \cos sx\, dx$.

Its inverse is defined by $f(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} F_c[f(x)] \cos sx\, ds.$

6) Find the Fourier Transform of $f(x) = \begin{cases} 0, & x < a \\ 1, & a < x < b \\ 0, & x > b \end{cases}$

Sol: $$F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx}\, dx.$$

$$= \frac{1}{\sqrt{2\pi}} \int_a^b e^{isx}\, dx = \frac{1}{\sqrt{2\pi}} \left[\frac{e^{isx}}{is} \right]_a^b = \frac{1}{\sqrt{2\pi}} \left[\frac{e^{isb} - e^{isa}}{2is} \right]$$

.

7) Find the Fourier transform of $e^{-|x|}$ **.**

Sol:

$$F(s) = F[e^{-|x|}] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-|x|} e^{isx}\, dx. = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-|x|} \cos sx dx + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-|x|} \sin sx dx..$$

$$= \sqrt{\frac{2}{\pi}} \int_0^{\infty} e^{-x} \cos sx dx. = \sqrt{\frac{2}{\pi}} \left[\frac{1}{1+s^2} \right]$$

8) Find the Fourier Cosine Transform of $f(x) = \begin{cases} x & 0 < x < 1 \\ 2 - x & 1 < x < 2 \\ 0 & x > 2 \end{cases}$

Sol: We know that

$$F_c[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos sx\, dx.$$

$$= \sqrt{\frac{2}{\pi}} \left[\int_0^1 x \cos sx\, dx + \int_1^2 (2 - x) \cos sx\, dx \right]$$

$$= \sqrt{\frac{2}{\pi}} \left[\left[x \left[\frac{\sin sx}{s} \right] + \frac{\cos sx}{s^2} \right]_0^1 + \left[(2 - x) \frac{\sin sx}{s} - \frac{\cos sx}{s^2} \right]_1^2 \right]$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{\sin s}{s} + \frac{\cos s}{s^2} - \frac{1}{s^2} + \frac{\cos s}{s^2} - \frac{\cos 2s}{s^2} - \frac{\sin s}{s} \right]$$

$$= \sqrt{\frac{2}{\pi}} \frac{1}{s^2} \left[2 \cos s - \cos 2s - 1 \right].$$

9) Find the Fourier cosine transform of $f(x) = \begin{cases} \cos x, & 0 < x < a \\ 0 & x \ge a \end{cases}$

Sol:

$$F_c(s) = \sqrt{\frac{2}{\pi}} \int_0^a \cos x \cos sx\, dx = \sqrt{\frac{2}{\pi}} \frac{1}{2} \int_0^a [\cos(s+1)x + \cos(s-1)x]\, dx$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{\sin(s+1)x}{s+1} + \frac{\sin(s-1)x}{s-1}\right]_0^a = \frac{1}{\sqrt{2\pi}}\left[\frac{\sin(s+1)a}{s+1} + \frac{\sin(s-1)a}{s-1}\right]$$

10) Find the Fourier cosine transform of $f(x) = \begin{cases} 1 & 0 < x < a \\ 0 & x > a \end{cases}$

Sol:

$$F_c(s) = \sqrt{\frac{2}{\pi}}\int_0^a \cos sx\,dx = \sqrt{\frac{2}{\pi}}\left[\frac{\sin sx}{s}\right]_0^a = \sqrt{\frac{2}{\pi}}\frac{\sin sa}{s}$$

11) Find the Fourier cosine Transform of e^{-ax}

,a>0. Sol: We know that

$$F_c[f(x)] = \int_0^\infty f(x)\cos sx\,dx..$$

$$F_c[e^{-ax}] = \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-ax}\cos sx\,dx.$$

$$= \sqrt{\frac{2}{\pi}}\left[\frac{a}{s^2+b^2}\right].$$

12) Find the Fourier cosine transform of $f(x) = x$ **.**

Sol: We know that

$$F_c[f(x)] = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos sx\,dx. \quad = \sqrt{\frac{2}{\pi}}\int_0^\infty x\cos sx\,dx$$

$$= \sqrt{\frac{2}{\pi}} R.P\int_0^\infty x e^{-isx}\,dx = \sqrt{\frac{2}{\pi}} R.P\left[x\frac{e^{-isx}}{(-is)} - \frac{e^{-isx}}{(-is)^2}\right]_0^\infty$$

$$\sqrt{\frac{2}{\pi}} R.P \left[\frac{-1}{s^2} \right]$$

$$= -\sqrt{\frac{2}{\pi}} \frac{1}{s^2}.$$

13) Find the Fourier Sine transform of $f(x) = e^{-x}$ **.**

Sol: We know that

$$F_s[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin sx dx = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-x} \sin sx dx = \frac{\sqrt{2}}{\pi} \left[\frac{s}{s^2 + 1} \right]$$

14) Find the Fourier cosine transform of $2e^{-3x} + 3e^{-2x}$ **.**

Sol: $F_c[2e^{-3x} + 3e^{-2x}] = \sqrt{\frac{2}{\pi}} \int_0^\infty (2e^{-3x} + 3e^{-2x}) \cos sx dx$

$$= \sqrt{\frac{2}{\pi}} \left[2\int_0^\infty e^{-3x} \cos sx dx + 3\int_0^\infty e^{-2x} \cos sx dx \right]$$

$$= \sqrt{\frac{2}{\pi}} \left[2\left(\frac{3}{s^2 + 9} \right) + 3\left(\frac{2}{s^2 + 4} \right) \right] = 6\sqrt{\frac{2}{\pi}} \left[\frac{1}{s^2 + 9} + \frac{1}{s^2 + 4} \right]$$

15) Find the Fourier cosine transform of $5e^{-2x} + 2e^{-5x}$ **.**

Sol: $F_c[5e^{-2x} + 2e^{-5x}] = \int_0^\infty (5e^{-2x} + 2e^{-5x}) \cos sx dx$

$$= \sqrt{\frac{2}{\pi}} \left[5\int_0^\infty e^{-2x} \cos sx dx + 2\int_0^\infty e^{-5x} \cos sx dx \right]$$

$$= \sqrt{\frac{2}{\pi}} \left[5\left(\frac{2}{s^2 + 4} \right) + 2\left(\frac{5}{s^2 + 25} \right) \right] = 10\sqrt{\frac{2}{\pi}} \left[\frac{1}{s^2 + 4} + \frac{1}{s^2 + 25} \right]$$

16) Find the Fourier cosine transform of $e^{-ax}\cos ax$

Sol: $F_c[e^{-ax}\cos ax] = \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-ax}\cos ax\cos sx\,dx$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{e^{-ax}}{2}[\cos(s+a)x + \cos(s-a)x]dx$$

$$= \frac{1}{\sqrt{2\pi}}\left\{\int_0^\infty e^{-ax}\cos(s+a)dx + \int_0^\infty e^{-ax}\cos(s-a)dx\right\}$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{a}{a^2+(s+a)^2} + \frac{a}{a^2+(s-a)^2}\right]$$

17) Find the Fourier cosine transform of $e^{-ax}\sin ax$

Sol: $F_c[e^{-ax}\sin ax] = \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-ax}\sin ax\cos sx\,dx$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{e^{-ax}}{2}[\sin(s+a)x - \sin(s-a)x]dx$$

$$= \frac{1}{\sqrt{2\pi}}\left\{\int_0^\infty e^{-ax}\sin(s+a)x\,dx - \int_0^\infty e^{-ax}\sin(s-a)x\,dx\right\}$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{s+a}{a^2+(s+a)^2} - \frac{s-a}{a^2+(s-a)^2}\right]$$

18) Find the Fourier Sine Transform of $f(x) = \begin{cases} x & 0 < x < 1 \\ 2 - x & 1 < x < 2 \\ 0 & x > 2 \end{cases}$

Sol: We know that

$$F_s[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin sx\, dx$$

$$= \sqrt{\frac{2}{\pi}} \left[\int_0^1 x \sin sx\, dx + \int_1^2 (2 - x) \sin sx\, dx \right]$$

$$= \sqrt{\frac{2}{\pi}} \left[\left[x\left(\frac{-\cos sx}{s}\right) + \frac{\sin sx}{s^2} \right]_0^1 + \left[(2 - x)\frac{-\sin sx}{s} - \frac{\sin sx}{s^2} \right]_1^2 \right]$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{-\cos s}{s} + \frac{\sin s}{s^2} - \frac{\sin 2s}{s^2} + \frac{\cos s}{s} + \frac{\sin s}{s^2} \right]$$

$$= \sqrt{\frac{2}{\pi}}\; 2\sin s \left[\frac{1 - \cos s}{s^2} \right].$$

19) Find the Fourier sine transform of $\frac{1}{x}$.

Sol: $F_s\left[\frac{1}{x}\right] = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\sin sx}{x}\, dx. = \sqrt{\frac{2}{\pi}} \cdot \frac{\pi}{2} = \sqrt{\frac{\pi}{2}} \quad \left[\int_0^\infty \frac{\sin mx}{x} dx = \frac{\pi}{2} \right]$

20) Find the Fourier sine transform of $3e^{-5x} + 5e^{-2x}$

Sol: $F_S[3e^{-5x} + 5e^{-2x}] = \sqrt{\frac{2}{\pi}} \int_0^\infty (3e^{-5x} + 5e^{-2x}) \sin sx\, dx$

$$= \sqrt{\frac{2}{\pi}}\left[\int_0^\infty 3e^{-5x}\sin sx\,dx + \int_0^\infty 5e^{-2x}\sin sx\,dx\right]$$

$$= \sqrt{\frac{2}{\pi}}\left[3\left[\frac{s}{s^2+25}\right] + 5\left[\frac{s}{s^2+4}\right]\right]$$

21) Find the Fourier sine transform of $7e^{-4x} + 4e^{-7x}$

Sol: $F_S[7e^{-4x} + 4e^{-7x}] = \sqrt{\frac{2}{\pi}}\int_0^\infty (7e^{-4x} + 4e^{-7x})\sin sx\,dx$

$$= \sqrt{\frac{2}{\pi}}\left[\int_0^\infty 7e^{-4x}\sin sx\,dx + \int_0^\infty 4e^{-7x}\sin sx\,dx\right]$$

$$= \sqrt{\frac{2}{\pi}}\left[7\left[\frac{s}{s^2+16}\right] + 4\left[\frac{s}{s^2+49}\right]\right]$$

22) Find the Fourier sine transform of $f(x) = \begin{cases} \sin x, & 0 < x < a \\ 0 & x \geq a \end{cases}$

Sol:

$$F_S(s) = \sqrt{\frac{2}{\pi}}\int_0^a \sin x \sin sx\,dx = \sqrt{\frac{2}{\pi}}\,\frac{1}{2}\int_0^a [\cos(s-1)x - \cos(s+1)x]\,dx$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{\sin(s-1)x}{s-1} - \frac{\sin(s+1)x}{s+1}\right]_0^a = \frac{1}{\sqrt{2\pi}}\left[\frac{\sin(s-1)a}{s-1} - \frac{\sin(s+1)a}{s+1}\right]$$

23) Solve the integral equation $\int_0^{\infty} f(x)\cos\lambda x dx = e^{-\lambda}$

Sol: Given $\int_0^{\infty} f(x)\cos\lambda x dx = e^{-\lambda}$.

$$\sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\cos\lambda x dx = \sqrt{\frac{2}{\pi}}\, e^{-\lambda}$$

$$F_c[f(x)] = \sqrt{\frac{2}{\pi}}\, e^{-\lambda}$$

$$\therefore f(x) = F_c^{-1}\left[\sqrt{\frac{2}{\pi}}\, e^{-\lambda}\right] = \sqrt{\frac{2}{\pi}}\int_0^{\infty}\sqrt{\frac{2}{\pi}}e^{-\lambda}\cos\lambda x d\lambda$$

$$\therefore f(x) = \frac{2}{\pi}\left(\frac{1}{1+x^2}\right)$$

24) State and prove change of scale property.

Sol: $$F[f(ax)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(ax)e^{isx}\, dx.$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)e^{i\frac{s}{a}t}\,\frac{dt}{a}$$ (by putting $t = ax$)

$$F[f(ax)] = \frac{1}{a}\, F\left(\frac{s}{a}\right) \text{ if } a > 0$$

Similarly $F[f(ax)] = -\dfrac{1}{\sqrt{2\pi}}\displaystyle\int_{-\infty}^{\infty} f(t)e^{i\frac{s}{a}t}\,\frac{dt}{a}$ if $a < 0$

$$F[f(ax)] = -\frac{1}{a}F\left(\frac{s}{a}\right) \text{ if } a<0$$

Hence $F[f(ax)] = \frac{1}{|a|}F\left(\frac{s}{a}\right)$

25) If $F_c[s]$ is the Fourier cosine transform of $f(x)$,prove that the Fourier cosine transform of $f(ax)$ is $\frac{1}{a}F_c\left(\frac{s}{a}\right)$.

Sol: $F_c[f(ax)] = \sqrt{\frac{2}{\pi}}\int_0^\infty f(ax)\cos sx\,dx$

Put $ax = t, dx = \frac{dt}{a}, t : 0 \to \infty.$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty f(t)\cos\left(\frac{s}{a}t\right)\frac{dt}{a} = \frac{1}{a}\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos\left(\frac{s}{a}x\right)dx$$

$$= \frac{1}{a}F_c\left(\frac{s}{a}\right).$$

26) If $F_S[s]$ is the Fourier cosine transform of $f(x)$, prove that the Fourier cosine transform of $f(ax)$ is $\frac{1}{a}F_S\left(\frac{s}{a}\right)$.

Sol: $F_S[f(ax)] = \sqrt{\frac{2}{\pi}}\int_0^\infty f(ax)\sin sx\,dx$

Put $ax = t, dx = \frac{dt}{a}$, $t : 0.$ to ∞

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty f(t)\sin\left(\frac{s}{a}t\right)\frac{dt}{a} = \frac{1}{a}\sqrt{\frac{2}{\pi}}\int_0^\infty f(t)\sin\frac{s}{a}t\,dt$$

$$=\frac{1}{a}F\left(\frac{s}{a}\right).$$

27) If $F(s)$ is the Fourier transform of $f(x)$, find the Fourier transform of $f(x-a)$.

Sol: $F[f(x-a)]=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{isx}f(x-a)dx$

Put $x-a=t,\ dx=dt,\ t:-\infty\to\infty$

$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{is(a+t)}.f(t)dt$$

$$F[f(x-a)]=e^{ias}F(s)$$

28) If $F(s)$ is the Fourier transform of $f(x)$,find the Fourier transform of $e^{iax}f(x)$.

Sol: $[e^{iax}f(x)]$

$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{iax}f(x)e^{isx}dx$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(x)e^{i(s+a)x}dx=F(s+a)$$

29) If $F(s)$ is the Fourier transform of $f(x)$, find the Fourier transform of $f(x+a)$.

Sol: $F[f(x+a)]=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{isx}f(x+a)dx$

Put $x+a=t,\ dx=dt, t:-\infty\to\infty$

$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{is(t-a)}.f(t)dt$$

$$F[f(x+a)] = e^{-ias} F(s)$$

30) If $F(s)$ is the Fourier transform of $f(x)$,find the Fourier transform of $e^{-iax} f(x)$.

Sol: $$F[e^{-iax} f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-iax} f(x) e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{i(s-a)x}\, dx = F(s-a)$$

31) If $F(s)$ is the Fourier transform of $f(x)$, derive the formula for the Fourier transform of $f(x)\cos ax$ in terms of F.

Sol:

$$F[f(x)\cos ax] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\cos ax\, e^{isx}\, dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\left[\frac{e^{iax}+e^{-iax}}{2}\right] e^{isx}\, dx$$

$$= \frac{1}{2}\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\left[e^{i(s+a)x} + e^{i(s-a)x}\right] dx$$

$$= \frac{1}{2}\left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{i(s+a)x}\, dx + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{i(s-a)x}\, dx\right]$$

$$= \frac{1}{2}[F(s+a) + F(s-a)]$$

32) If $F_S(s)$ is the Fourier sine transform of $f(x)$,show that

$$F_S[f(x)\cos ax] = \frac{1}{2}[F_S(s+a) + F_S(s-a)]$$

Sol:

$$F_S[f(x)\cos ax] = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos ax \sin sx\,dx$$

$$= \frac{1}{2}\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)[\sin(s+a)x + \sin(s-a)x]dx$$

$$= \frac{1}{2}\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin(s+a)x\,dx + \frac{1}{2}\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin(s-a)x\,dx\,.$$

$$= \frac{1}{2}[F_S(s+a) + F_S(s-a)]$$

33) If $F_c(s)$ is the Fourier cosine transform of $f(x)$,show that

$$F_c[f(x)\cos ax] = \frac{1}{2}[F_c(s+a) + F_c(s-a)]$$

Sol:

$$F_c[f(x)\cos ax] = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos ax \cos sx\,dx$$

$$= \frac{1}{2}\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)[\cos(s+a)x + \cos(s-a)x]dx$$

$$= \frac{1}{2}\left[\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos(s+a)x\,dx + \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos(s-a)x\,dx\right].$$

$$= \frac{1}{2}[F_c(s+a) + F_c(s-a)]$$

34) If $F_S(s)$ is the Fourier sine transform of $f(x)$,show that

$$F_S[f(x)\sin ax] = \frac{1}{2}[F_c(a-s) - F_c(s+a)]$$

Sol:

$$F_S[f(x)\sin ax] = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin ax \sin sx\,dx$$

$$= \frac{1}{2}\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)[\cos(a-s)x - \cos(a+s)x]dx$$

$$= \frac{1}{2}\left[\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos(a-s)x\,dx - \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos(s+a)x\,dx\right].$$

$$= \frac{1}{2}[F_c(a-s) - F_c(s+a)]$$

35) If $F_s(s)$ is the Fourier sine transform of $f(x)$,show that

$$F_c[f(x)\sin ax] = \frac{1}{2}[F_S(s+a) + F_S(s-a)]$$

Sol: $$F_c[f(x)\sin ax] = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin ax \cos sx\,dx$$

$$= \frac{1}{2}\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)[\sin(a+s)x + \sin(a-s)x]dx$$

$$= \frac{1}{2}\left[\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin(a+s)x\,dx + \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin(a-s)x\,dx\right].$$

$$= \frac{1}{2}[F_S(a-s) + F_S(s+a)]$$

36) Find $F[x^n f(x)]$ **and** $F\left[\frac{d^n f(x)}{dx^n}\right]$ **in terms of Fourier transform of** $f(x)$.

Sol: $F[x^n f(x)] = (-i)^n \frac{d^n}{ds^n} [F(s)]$

$F\left[\frac{d^n}{dx^n} f(x)\right] = (-is)^n F(s)$ where $F(s) = F[f(x)]$.

37) Prove that $F_S[xf(x)] = -\frac{d}{ds}[F_c(s)]$

Sol: We know that $F_c[f(x)] = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos sx\,dx$.

Diff both sides w.r.t s

$$\frac{d}{ds}F_c[f(x)] = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\frac{\partial}{\partial s}(\cos sx)dx = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)(-x\sin sx)dx$$

$$= -\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)x\sin sx\,dx = -F_S[xf(x)]$$

(i.e) $F_S[xf(x)] = -\frac{d}{ds}[F_c(s)]$

38) Show that $F_c[xf(x)] = \frac{d}{ds}F_S[s]$

Sol: $F_S[s] = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin sx\,dx$

Diff both sides w.r.t s,

$$\frac{d}{ds}F_S(s) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)(x\cos sx)dx = F_c[xf(x)]$$

39) Find the Fourier cosine transform of xe^{-ax}

Sol: $F_c[xe^{-ax}] = \frac{d}{ds} F_S[e^{-ax}] = \frac{d}{ds}\left[\sqrt{\frac{2}{\pi}} \frac{s}{s^2+a^2}\right] = \sqrt{\frac{2}{\pi}}\left[\frac{a^2-s^2}{(s^2+a^2)^2}\right]$

$\left[\text{since } F_S[e^{-ax}] = \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-ax}\sin sx\,dx = \sqrt{\frac{2}{\pi}}\left(\frac{s}{s^2+a^2}\right)\right]$

40) Find the Fourier sine transform of xe^{-ax}

Sol: $F_S[xe^{-ax}] = -\frac{d}{ds} F_S[e^{-ax}] = -\frac{d}{ds}\left[\sqrt{\frac{2}{\pi}} \frac{s}{s^2+a^2}\right] = \sqrt{\frac{2}{\pi}}\left[\frac{2as}{(s^2+a^2)^2}\right]$

$\left[\text{since } F_c[e^{-ax}] = \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-ax}\cos sx\,dx = \sqrt{\frac{2}{\pi}}\left(\frac{a}{s^2+a^2}\right)\right]$

41) State the Parseval's identity for Fourier Transform.

Sol: $\int_{-\infty}^{\infty} |F(s)|^2\,ds = \int_{-\infty}^{\infty} |f(x)|^2\,dx$ where $F(s) = F[f(x)]$

42) State the Parsevals identity for Fourier sine and cosine transform.

Sol: $\int_0^\infty |F_c(s)|^2\,ds = \int_0^\infty |f(x)|^2\,dx$ where $F_c(s) = F_c[f(x)]$.

$\int_0^\infty |F_S(s)|^2\,ds = \int_0^\infty |f(x)|^2\,dx$ where $F_S(s) = F_S[f(x)]$

43) State the Convolution theorem for Fourier transforms

Sol: If $F[f(x)] = F(s)$ and $F[g(x)] = G(s)$

Then $F[f(x) * g(x)] = F(s).G(s)$ where $f * g = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)g(x-t)dt$

44) Find the Fourier transform of $e^{-a|x|}$

Sol: We know that $F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)e^{isx}\, dx.$

$$F[e^{-a|x|}] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-a|x|} e^{isx}\, dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-a|x|} (\cos sx + i \sin sx) dx$$

$$= \frac{2}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-ax} \cos sx dx = \sqrt{\frac{2}{\pi}} \left[\frac{a}{s^2 + a^2} \right]$$

45) If $e^{-x} = \frac{2}{\pi} \int_0^{\infty} \frac{s}{s^2+1} \sin sx ds$ then show that $\int_0^{\infty} \frac{x \sin mx}{1+m} dx = \frac{\pi}{2} e^{-m}$.

Sol: Given $e^{-x} = \frac{2}{\pi} \int_0^{\infty} \frac{s}{s^2+1} \sin sx ds$ that is $\frac{\pi}{2} e^{-x} = \int_0^{\infty} \frac{s}{s^2+1} \sin sx ds$.

Put $x = m$,we get $\frac{\pi}{2} e^{-m} = \int_0^{\infty} \frac{s}{s^2+1} \sin sm ds = \int_0^{\infty} \frac{x}{x^2+1} \sin mx dx$.

46) Prove that $F[f(-x)] = F(-s)$

Sol: $F[f(-x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(-x)e^{isx}\, dx$

Put $-x = y$ where dx $x = -\infty, y = \infty$

$= -dy$ where $x = \infty, y = -\infty$

$$= \frac{1}{\sqrt{2\pi}} \int_{\infty}^{-\infty} f(y) e^{-isy} (-dy) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(y) e^{-isy} dy$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-isx} dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{i(-s)x} dx = F(-s)$$

47) Prove that $F[x^n f(x)] = (-i)^n \frac{d^n F}{ds^n}$.

Sol: $F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} dx$

$$\frac{d}{ds} F(s) = \frac{d}{ds} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} (ix) dx.$$

$$\frac{d^2}{ds^2} F(s) = \frac{d}{ds} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} (ix) dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} (ix)^2 dx.$$

In general $\frac{d^n}{ds^n} F(s) = (i)^n \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^n f(x) e^{isx} dx = (i)^n F[x^n f(x)].$

Hence $F[x^n f(x)] = (-i)^n \frac{d^n}{ds^n} F(s).$

48) Prove that $F_S(f'(x)) = -sF(s)$

Sol: $F_S(f'(x)) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f'(x) \sin sx dx = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \sin sx d(f(x))$

$$= \sqrt{\frac{2}{\pi}} \left[\left[\sin sx f(x)\right]_0^{\infty} - \int_0^{\infty} f(x) s \cos sx dx \right]$$

$$= \sqrt{\frac{2}{\pi}} \left[-s \int_0^{\infty} f(x) \cos sx dx \right] = -sF(s)$$

49) Give the function which is self reciprocal under fourier sine and cosine transform.

Ans: $f(x) = \frac{1}{\sqrt{x}}$

50) State modulation theorem in Fourier transforms. Ans:

If $F(s) = F[f(x)]$, then $F[f(x)\cos ax] = \frac{1}{2}[F(s+a) + F(s-a)]$

51) Find the fourier sine transform of $\frac{1}{x}$.

Sol: $F_S[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin sx dx$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{1}{x} \sin sx dx$$

From complex analysis, $\int_0^\infty \frac{1}{x} \sin sx dx = \frac{\pi}{2}$ (contour integration over semi circle)

$$\therefore F_S\left[\frac{1}{x}\right] = \sqrt{\frac{2}{\pi}} \left(\frac{\pi}{2}\right) = \sqrt{\frac{\pi}{2}}$$

52) State Parseval's identity on complex fourier transform.

Sol: $\int_{-\infty}^{\infty} |F(s)|^2 ds = \int_{-\infty}^{\infty} |f(x)|^2 dx$ where $F(s) = F[f(x)]$

53) Find the finite fourier sine transform of $f(x) = 2x \quad 0 < x < 4$.

Sol: $F_S[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin sx dx$

$$= \sqrt{\frac{2}{\pi}} \int_0^4 2x \sin sx \, dx$$

$$= 2\sqrt{\frac{2}{\pi}} \left[\frac{-x \cos sx}{s} + \frac{\sin sx}{s^2} \right]_0^4$$

$$F_s[f(x)] = 2\sqrt{\frac{2}{\pi}} \left[\frac{-4 \cos 4s}{s} + \frac{\sin 4s}{s^2} \right]$$

54) If $F\{f(x)\} = F(s)$ **, prove that** $F\{x^2 f(x)\} = -\frac{d^2}{ds^2} F(s)$

Sol: Since, $F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} \, dx$

$$\frac{d}{ds} F(s) = \frac{d}{ds} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} \, dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} (ix) dx.$$

$$\frac{d^2}{ds^2} F(s) = \frac{d}{ds} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} (ix) dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} (ix)^2 \, dx = -\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(x^2 f(x) \right) e^{isx} \, dx = F\{x^2 f(x)\}$$

Hence $F\{x^2 f(x)\} = -\frac{d^2}{ds^2} F(s)$.

55) Find the Fourier cosine transform of $f(x) = \begin{cases} x, & 0 < x < \pi \\ 0, & x \geq \pi \end{cases}$

Sol:

$$F_c[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \cos sx dx = \sqrt{\frac{2}{\pi}} \int_0^{\pi} x \cos sx dx$$

$$= \sqrt{\frac{2}{\pi}}\left[\frac{x \sin sx}{s} + \frac{\cos sx}{s^2}\right]_0^{\pi} = \sqrt{\frac{2}{\pi}}\left[\frac{\cos s\pi}{s^2} - \frac{1}{s^2}\right]$$

UNIT-IV Applications of Partial Differential Equations

Two Marks Questions And Answers

1. Classify : $\frac{\partial^2 u}{\partial x^2} + 2\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial u}{\partial x} - \frac{\partial u}{\partial y} = 0$

Solution:

Here, $A = 1,\ B = 2, C = 1$

$$\therefore B^2 - 4AC = 4 - 4 = 0$$

$\therefore$ The given P.D.E is parabolic.

2. Classify: $2f_{xx} + f_{xy} - f_{yy} + f_x + 3f_y = 0$

Solution:

Here, $A = 2,\ B = 1, C = -1$

$$\therefore B^2 - 4AC = 1 - 4(2)(-1) = 1 + 8 = 9 > 0$$

$\therefore$ The given P.D.E is hyperbolic.

3. Classify: $U_{xx} + 3U_{xy} + 4U_{yy} + U_x - 3U_y = 0$

Solution:

Here, $A = 1,\ B = 3, C = 4$

$$B^2 - 4AC = 9 - 4(1)(4) = 9 - 16 = -7 < 0$$

$\therefore$ The given P.D.E is elliptic.

4. Classify: $x^2 \frac{\partial_2 u}{\partial x^2 \partial y^2} + (1-y^2)\frac{\partial_2 u}{} = 0, -\infty < x < \infty, -1 < y < 1$

Solution:

$$A = x^2,\ B = 0, C = (1-y^2)$$
$$\therefore B^2 - 4AC = 0 - 4x^2(1-y^2) = -4x^2(1-y^2) = 4x^2(1-y^2)$$
$$\therefore B^2 - 4AC = 4x^2(1-y^2) > 0,$$
$$\{\ x^2 > 0 and -1 < y < 1 \Rightarrow y^2 < 1 \therefore 1-y^2 > 0\}$$

$\therefore$ The given P.D.E is elliptic.

5 Classify: $(1+x^2)\frac{\partial^2 u}{\partial x^2} + (5+2x^2)\frac{\partial^2 u}{\partial x \partial t} + (4+x^2)\frac{\partial^2 u}{\partial t^2} = 0$

Solution:

$$A = 1+x^2,\ B = 5+2x^2, C = 4+x^2$$
$$\therefore B^2 - 4AC = (5+2x^2)^2 - 4(1+x^2)(4+x^2)$$
$$= 25 + 20x^2 + 4x^4 - 16 - 20x^2 + 4x^4 = 9 > 0$$

$\therefore$ The given P.D.E is hyperbolic.

6. Classify: $U_{xx} + yU_{xy} + \frac{x}{4}U_{yy} - U_x + U = 0$.

Solution:

$$A = 1,\ B = y, C = \frac{x}{4}$$
$$B^2 - 4AC = y^2 - 4(1)\left(\frac{x}{4}\right) = y^2 - x$$
$$\therefore B^2 - 4AC = y^2 - x > 0, \quad if \quad y^2 > x \Rightarrow hyperbolic$$

$$= 0, \quad if \ \ y^2 = x \Rightarrow parabolic$$

$$< 0, \ if \quad y^2 < x \Rightarrow elliptic$$

7. Classify: $xf_{xx} + yf_{yy} = 0, x > 0, y > 0$

Solution:

$$A = x,\ B = 0,\ C = y$$
$$\therefore B^2 - 4AC = 0 - 4xy = -4xy < 0 \left\{ \therefore \ x > 0 and y > 0 \right\}$$

$\Rightarrow$ Given P.D.E is elliptic.

8. Classify the following P.D.E $y^2U_{xx} - 2xyU_{xy} + x^2U_{yy} + 2U_x - 3U = 0$

Solution:

$$A = y^2,\ B = -2xy, C = x^2$$
$$\therefore B^2 - 4AC = 4x^2y^2 - 4y^2x^2 = 0$$

$\Rightarrow$ The given P.D.E is parabolic.

9. Classify: $y^2U_{xx} + U_{yy} + U_x^2 + U_y^2 + 7 = 0$

Solution:

$$A = y^2,\ B = 0,\ c = 1$$
$$\therefore B^2 - 4AC = 0 - 4y^2 = -4y^2 < 0$$

$\Rightarrow$The given P.D.E is elliptic.

10. (a)Classify: $xf_{xx} + f_{yy} = 0$

Solution:

$$A = x,\ B = 0, C = 1$$
$$\therefore B^2 - 4AC = 0 - 4x = -4x < 0, if x > 0 \Rightarrow elliptic$$

$$= 0, if x = 0 \Rightarrow parabolic$$

$$> 0, if x < 0 \Rightarrow hyperbolic.$$

(b). Classify: $3\dfrac{\partial^2 u}{\partial x^2} + 4\dfrac{\partial^2 u}{\partial x \partial y} + 6\dfrac{\partial^2 u}{\partial y^2} - 2\dfrac{\partial u}{\partial y} - u = 0$

11. State one –dimensional wave equation.

The one-dimensional wave equation is

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}$$

12. What is the constant a^2 in the wave equation $u_{tt} = a^2 u_{xx}$?

Solution:

$a^2 = \dfrac{T}{m}$ Where "T" is the tension caused by stretching the string before fixing it at the end points, and "m" is the mass per unit length of the string.

13. State the possible solutions of one dimensional wave equation. Solution:

The possible solutions of one dimensional wave equation are

$$y(x,t) = \left(Ae^{px} + Be^{-px}\right)\left(Ce^{pat} + De^{-pat}\right)$$
$$y(x,t) = \left(A\cos px + B\sin px\right)\left(C\cos pat + D\sin pat\right)$$
$$y(x,t) = \left(Ax + B\right)\left(Cx + D\right)$$

14. State the assumptions made in the derivation of one dimensional wave equation. Solution:

(1)The mass of the string per unit length is constant.

(2)The string is perfectly elastic and does not offer any resistance to bending

3)The tension "T" caused by stretching the string before fixing it at the end points is constant at all points of the deflected string and at all times.

(4) "T" is so large that other external forces such as weight of the string and friction may be considered negligible.

(5)Deflection "y" and the slope $\frac{\partial y}{\partial x}$ at every point of the string are small, so that their higher powers may be neglected.

15. Write the boundary conditions and initial conditions for solving the vibration of string equation, if the string (of length "l") is subjected initial displacement $f(x)$

Solution:

Boundary conditions:

$$y(0, t) = 0$$
$$y(l, t) = 0$$

Initial conditions:

$$\left(\frac{\partial y}{\partial t}\right)_{t=0} = 0$$

$y(x,0) = f(x)$ =initial development.

16. Write the boundary conditions and initial conditions for solving the vibration of string equation, if the string (of length l) is subjected to the initial velocity $g(x)$

Solution:

Boundary conditions:

$$y(0, t) = 0$$
$$y(l, t) = 0$$

Initial conditions:

$$y(x,0) = 0$$
$$\left(\frac{\partial y}{\partial t}\right)_{t=0} = g(x)$$

17. Write the boundary conditions and initial conditions for solving the vibration of string equation, if the string is subjected to initial displacement $f(x)$ and initial velocity $g(x)$

Solution:

Boundary conditions:

$$y(0,t) = 0$$
$$y(l,t) = 0$$

Initial conditions:

$$\left(\frac{\partial y}{\partial t}\right)_{t=0} = g(x)$$
$$y(x,0) = f(x)$$

18. A tightly stretched string of length "2l" is fastened at both ends. The mid point of the string is displaced by a distance "b" transversely and the string is released from rest in this position. Write the boundary and initial conditions.

Solution:

$$O(0,0),\ M = (l,b),\ A = (2l,0)$$

$lineOM$:

$$\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1}$$
$$\Rightarrow \frac{y-0}{b-0} = \frac{x-0}{l-0}$$
$$\Rightarrow \frac{y}{b} = \frac{x}{l}$$
$$\Rightarrow y = \frac{bx}{l}$$

$lineMA$:

$$\frac{y-b}{a-b} = \frac{x-l}{2l-l}$$
$$\Rightarrow \frac{y-b}{-b} = \frac{x-l}{l}$$
$$\Rightarrow y - b = -\frac{b(x-l)}{l}$$
$$\Rightarrow y = \frac{bl - bx + bl}{l} = \frac{b(2l-x)}{l}$$

Boundary conditions:

$$y(0,t) = 0$$
$$y(2l,t) = 0$$

Initial conditions:

$$\left(\frac{\partial y}{\partial t}\right)_{t=0} = 0$$

$$y(x,0) = \begin{cases} \frac{bx}{l}, & 0 < x < l \\ \frac{b(2l-x)}{l}, & l < x < 2l \end{cases}$$

19. A uniform string of length "l" is fastened at both ends. The string is at rest, with the point $x = b$ drawn aside through a small distance "d" and released at time t=0.Write the initial conditions,

Solution:

$o(0,0), A = (l,0)$

$lineOB$

$$\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1}$$

$$\Rightarrow \frac{y - 0}{d - 0} = \frac{x - 0}{b - 0} \Rightarrow \frac{y}{d} = \frac{x}{b} \Rightarrow y = \frac{d}{b} x$$

$lineBA$

$$\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1} \Rightarrow \frac{y - d}{0 - d} = \frac{x - b}{l - b}$$

$$\Rightarrow y - d = \frac{-d(x - b)}{l - b}$$

$$y = \frac{dl - db - dx + db}{l - b}$$

$$y = \frac{d(l - x)}{l - b}$$

$\therefore Initialconditions$

$$\left(\frac{\partial y}{\partial t}\right)_{t=0} = 0$$

$$y(x,0) = \begin{cases} \frac{d}{b} x, & 0 < x < b \\ \frac{d(l - x)}{l - b}, & b < x < l \end{cases}$$

$LineOA$

$$\frac{y - y_1}{y - y} = \frac{x - x_1}{x - x}$$

$$\Rightarrow \frac{y-0}{1-0} = \frac{x-0}{10-0} \Rightarrow \frac{y}{1} = \frac{x}{10} \Rightarrow y = \frac{x}{10}$$

20. The points of trisection of a tightly stretched string of length 30 cm with fixed ends pulled aside through a distance of 1cm on opposite sides of the position of equilibrium and the string is released from rest. Write the boundary and initial conditions.

Solution:

$$O(0,0),\ A(10.1),\ B(20,-1), C(30,0)$$

$$Line AB$$

$$\frac{y-1}{-1-1} = \frac{x-10}{20-10} \Rightarrow \frac{y-1}{-2} = \frac{x-10}{10} \Rightarrow y-1 = \frac{x-10}{-5}$$

$$\Rightarrow y = 1 - \left(\frac{x-10}{5}\right)$$

$$y = \frac{5-x+10}{5} = \frac{15-x}{5}$$

$$Line BC$$

$$\frac{y+1}{0+1} = \frac{x-20}{30-10}$$

$$y+1 = \frac{x-20}{10}$$

$$y = -1 + \frac{x-20}{10} = \frac{x-30}{10}$$

$$Initial conditions:$$

$$\left(\frac{\partial y}{\partial t}\right)_{t=0} = 0,$$

$$y(0,t) = \begin{cases} \frac{x}{10}, 0 \le x \le 10 \\ \frac{15-x}{5}, 10 \le x \le 20 \\ \frac{x-30}{10}, 20 \le x \le 30 \end{cases}$$

21. State one dimensional heat flow equation.

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$$

22. What does "α^2" denote in one dimensional heat flow equation.

$\alpha^2 = \dfrac{k}{c\rho}$ Where k - Thermal conductivity

ρ - Density

c - Specific heat

23. State three possible solutions of one dimensional heat flow equation.

$$u(x,t) = \left(Ae^{px} + Be^{-px}\right)e^{p^2\alpha^2 t}$$
$$u(x,t) = \left(A\cos px + B\sin px\right)e^{-p^2\alpha^2 t}$$
$$u(x,t) = Ax + B$$

24. State two laws used in the derivation of one dimensional heat flow

equation. Laws of thermodynamics:

1. Increase in heat in the element in Δt time
   = (specific heat) (mass of the element) (Increase in temperature)

2. The rate of flow of heat across any area "A" is proportional to A and the temperature

gradient normal to the area, (i.e.) $\dfrac{\partial u}{\partial x}$. Where the constant of the

Proportionality is the thermal conductivity.

25. A uniform rod of length 50 cm with insulated sides is initially at a uniform temperature $100°c$ .Its ends are kept at $0°c$ write the boundary and initially conditions.

Solution:

Boundary conditions:

$$u(0,t) = 0$$
$$u(50,t) = 0$$

Initial conditions:

$$u(x,0) = 100°c$$

26. Write the boundary and initial conditions in a homogeneous bar of length π which is insulated laterally, if the ends are kept at zero temperature and if, initially, the temperature is "k" at the centre of the bar and fully uniformly to zero at its ends.

Solution:

$$o(0,0) : A = \left(\frac{\pi}{2}, k\right) : B = (\pi, 0)$$

$$The line OA$$

$$\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1}$$

$$\Rightarrow \frac{y - 0}{k - 0} = \frac{x - 0}{\frac{\pi}{2} - 0} \Rightarrow \frac{y}{k} = \frac{x}{\frac{\pi}{2}}$$

$$\Rightarrow y = \frac{2kx}{\pi}$$

ThelineAB

$$\frac{y-y_1}{y_2-y_1}=\frac{x-x_1}{x_2-x_1} \Rightarrow \frac{y-k}{0-k}=\frac{x-\frac{\pi}{2}}{\pi-\frac{\pi}{2}}$$

$$\Rightarrow \frac{y-k}{-k}=\frac{x-\frac{\pi}{2}}{\frac{\pi}{2}} \Rightarrow y-k=\frac{-2k}{\pi}\left(x-\frac{\pi}{2}\right)$$

$$\Rightarrow y=\frac{k\pi-2k\,x+k\pi}{\pi}$$

$$\Rightarrow y=\frac{2k\,(\pi-x)}{\pi}$$

Boundary conditions:

$$u(0,t)=0$$
$$u(\pi,t)=0$$

Initial conditions:

$$u(x,0)=\begin{cases}\frac{2kx}{\pi}, & 0<x<\frac{\pi}{2}\\ \frac{2k(\pi-x)}{\pi}, & \frac{\pi}{2}<x<\pi\end{cases}$$

27. A rod of length 20 cm has its ends A and B kept at $30°c$ and $90°c$ respectively, until steady state conditions prevail. Find the steady state solution.

Solution:

Boundary conditions:

$$u(0,\,t)=30° ----(1)$$
$$u(20,\,t)=90° ----(2)$$

The solution of steady state is

$$u(x,t) = ax + b ----(3)$$

Applying (1) in (3),

$$u(0) = b = 30^\circ$$
$$\Rightarrow b = 30^\circ$$

Applying (2) in (3),

$$u(20) = 20a + b = 90^\circ$$
$$\Rightarrow 20a + b = 90^\circ$$
$$\Rightarrow 20a + 30^\circ = 90^\circ$$
$$\Rightarrow 20a = 60^\circ$$
$$\Rightarrow a = 3$$

Substitute “a” & “b” in (3),

$$u(x) = 3x + 30^\circ$$

28. An insulated rod of length 60 cm has its ends at A and B maintained at $20^\circ c$ and $80^\circ c$ respectively. Find the steady state solution of the rod.

Solution:

One dimensional heat flow equation is

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2} ----(1)$$

Boundary condition:

$$u(0,t) = 20^\circ ----(2)$$
$$u(60,t) = 80^\circ ----(3)$$

The steady state solution of equation (1) is

$$u(x) = ax + b ----(4)$$

Applying (2) in (4), we get

$$u(0) = 20 \Rightarrow a(0) + b = 20$$
$$\Rightarrow b = 20$$

Applying (3) in (4), we get

$$u(60) = 80° \Rightarrow a(60) + b = 80°$$
$$\Rightarrow 60a + 20 = 80$$
$$\Rightarrow 60a = 60$$
$$\Rightarrow a = 1$$

Substitute “a” and “b” in (4), we get

$$u(x) = 1.x + 20°$$
$$= x + 20°$$

29. When the ends of a rod length 20 cm are maintain at the temperature $10°c$ and $20°c$ respectively until steady state is prevailed. Determine the steady state temperature of the rod.

Solutio:

The steady state solution is

$$u(x, t) = ax + b ----(1)$$

The boundary conditions are

$$u(0) = 10° ----(2)$$
$$u(20) = 20° ----(3)$$

Applying (2) in (1), we get

$$u(0) = a.0 + b = 10$$
$$b = 10$$

Applying (3) in (1), we get

$$u(20) = a.20 + b = 20$$
$$\Rightarrow 20a + 10 = 20$$
$$\Rightarrow 20a = 10$$
$$\Rightarrow a = \frac{10}{20}$$
$$\Rightarrow a = \frac{1}{2}$$

Substitute "a" & "b" in (1)

$$u = \frac{x}{2} + 10$$

30. What is the basic difference between the solutions of one dimensional wave equation and one dimensional heat equation?

Solution:

The suitable solution $y(x,t) = (A\cos px + B\sin px)(C\cos pat + D\sin pat)$ of one dimensional wave equation is periodic in nature. But the solution

$u(x,t) = (A\cos px + B\sin px)e^{-\alpha^2 p^2 t}$ of one dimensional heat flow equation is not periodic in nature.

31. Write the steady state two dimensional heat flow equation in Cartesian co-ordinates. Solution:

The two dimensional heat flow equation is

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} - - - (1)$$

32. Write the possible solutions of two dimensional heat flow equation.

The possible solutions are

$$u(x, y) = \left(Ae^{px} + Be^{-px}\right)\left(C\cos py + D\sin py\right)$$
$$u(x, y) = \left(A\cos px + B\sin px\right)\left(Ce^{py} + De^{-py}\right)$$
$$u(x, y) = (Ax + B)(Cx + D)$$

33. A rectangular plate with insulated surfaces is 8 cm wide and so long compared to its width that it may be considered infinite in length. If the temperature along one short edge y=0 is kept at $u(x,0) = 100\sin\left(\frac{\pi x}{8}\right), 0 < x < 8$**, while the other two long edges** $x = 0$ **and** $x = 8$ **as well as short edges are kept at** $0°c$**, write the boundary conditions.**

Solution:

The Boundary conditions are

$$u(0, y) = 0$$
$$u(8, y) = 0$$
$$\lim_{y\to\infty} u(x, y) = 0$$
$$u(x,0) = 100\sin\frac{\pi x}{8}$$

34. A rectangular plate with insulated surfaces is 10 cm wide and so long compared to its width that it may be considered infinite in length. The temperature along short edge

$$u(x,0) = \begin{cases} 20x, 0 < x < 5 \\ 20(10 - x), 5 < x < 10 \end{cases}$$

Other edges are kept at $0°c$. Write the boundary conditions.

Solution:

Boundary conditions:

$$u(0, y) = 0$$
$$u(10, y) = 0$$
$$\underset{y \to \infty}{lt}\ u(x, y) = 0$$

$$u(x,0) = \begin{cases} 20x, 0 < x < 5 \\ 20(10 - x), 5 < x < 10 \end{cases}$$

35. An infinitely long rectangular plate with insulated surface is 10 cm wide. The two long edges are one short edge are kept at $0°c$ while the other short edge x=0 is kept at

$$u = \begin{cases} 20\,y; 0 < x < 5 \\ 20(10 - y); 5 < y < 10 \end{cases}$$

Solution:

Boundary conditions:

$$u(x,0) = 0$$
$$u(x,10) = 0$$
$$\underset{x \to \infty}{lt}\ u(x, y) = 0$$

$$u(0, y) = u = \begin{cases} 20y, 0 < y < 5 \\ 20(10 - y), 5 < y < 10 \end{cases}$$

36. A square plate is bounded by the lines $x = 0, y = 0, x = 20, y = 20.$ its faces are insulated. The temperature along the upper horizontal edge is given by

$u(x,20) = x(20 - x), 0 < x < 20$ **While the other three edges are kept at $0°c$ .Write the Boundary conditions.**

Solution:

Boundary conditions are

$$u(0, y) = 0$$
$$u(20, y) = 0$$
$$u(x,0) = 0$$
$$u(x,20) = x(20 - x)$$

37. A square plane of side 30 cm is bounded by $x = 0, x = 30, y = 30$.the edges $x = 30, y = 0$ are kept $50°c$ and $x = 0, y = 20$ are kept at $0°c$, Formulate the problem.

Solution:

Boundary conditions:

$$u(0, y) = 0 \; u(30, y) = 0 \; u(x,0) = 50° \; u(x,30°) = 0°$$

38. A rectangular plate with insulated surfaces is a cm wide and so long compared to its width that it may be considered infinite in length without introducing an appreciable error. If the two long edges $x = 0$ and $x = a$ and the short edge at infinity are kept at temperature $0°c$, while the other short edge $y = 0$ is kept at temperature "T" (constant) Write the Boundary conditions.

Solution:

Boundary conditions:

$$u(0, y) = 0$$
$$u(a, y) = 0$$
$$u(x, \infty) = 0$$
$$u(x,0) = T$$

39. An infinitely long plane uniform plate by two parallel edges and end right angles to them. The breadth is π, and this end is maintained at a temperature u_0

and other edges are kept at $0°c$. Write the Boundary conditions:

Solution:

Boundary conditions:

$$u(0, y) = 0$$
$$u(\pi , y) = 0$$
$$u(x, \infty) = 0$$
$$u(x,0) = u_0$$

40. An infinitely long plate in the form of an area is enclosed between the lines $y = 0 and y = \pi$ for positive values of x .The temperature is zero along the edges $x = 0$ is kept at temperature $k\,y$, Find the steady state temperature distribution in the plate.

Solution:

Boundary conditions:

$$u(x,0) = 0$$
$$u(x, \pi) = 0$$
$$u(\infty, y) = 0$$
$$u(0, y) = k\,y$$

41. A square plate of length 20 cm has its faces insulated and its edges along
$x = 0, x = 20, y = 0 \,\&\, y = 20$ **.If the temperature along the edge** **is given by**

$$x = 20$$

$$u = \begin{cases} \frac{T}{10} y; 0 \le y \le 10 \\ \frac{T}{20}(20 - y); 10 \le y \le 20 \end{cases}$$

While the other three edges are kept at $0°c$ **, write the Boundary conditions.**

Solution:

$$u(x,0) = 0$$
$$u(x,20) = 0$$
$$u(0, y) = 0$$

$$u(20, y) = \begin{cases} \frac{T}{10} y; 0 \le y \le 10 \\ \frac{T}{20}(20 - y); 10 \le y \le 20 \end{cases}$$

42. A rectangular plate is bounded by the lines $x = 0, x = a, y = 0 \,\&\, y = b$ **.Its surfaces are insulated. The temperatures along** $x = 0, y = 0$ **are kept at** $0°c$ **and others at** $100°c$ **. Write the boundary conditions.**

Solution:

Boundary conditions:

$$u(0, y) = 0$$
$$u(a, y) = 100°c$$
$$u(x,0) = 0$$
$$u(x, b) = 100°c$$

43. Classify: $u_{xx} = u_{yy}$

Solution:

$$A = 1, B = 0, C = -1$$
$$\therefore B^2 - 4AC = 0 - 4(1)(-1) = 4 > 0$$

$\therefore$ The given P.D.E is hyperbolic.

44. (a)Write the general solution of $y(x, t)$ of vibrating motion of a string of length "l" with fixed end points and zero initial velocity.

Solution:

$$y(x,t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)\cos\left(\frac{n\pi ct}{l}\right)$$

44.(b) Write the general solution of $y(x, t)$ of vibrating motion of length "l" and fixed end points and zero initial shape.

Solution:

$$y(x, t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)\sin\left(\frac{n\pi ct}{l}\right) \quad y(x,t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)\sin\left(\frac{n\pi ct}{l}\right)$$

45. Classify the P.D.E $xf_{xx} + yf_{yy} = 0$

Solution:

$$A = x, B = 0, C = y$$

$B^2 - 4AC = 0 - 4xy = -4xy < 0$ If $\begin{bmatrix} x < 0 \ \& \ y < 0 \\ x > 0 \ \& \ y > 0 \end{bmatrix}$

$= 0$ If $x = 0$ or $y = 0$

> 0 If $\begin{bmatrix} x > 0 \ \& \ y < 0 \\ or \\ x < 0 \ \& \ y > 0 \end{bmatrix}$

$\therefore$ The given P.D.E is elliptic.

If
$$(1)\{x > 0 \ \& \ y > 0\}(or)\{x < 0 \ \& \ y < 0\}$$
$$(2) x = 0 (or)\ y = 0$$
$$(3)\{x > 0 \ \& \ y < 0\}(or)\{x < 0 \ \& \ y > 0\}$$

46. The ends A and B of a rod of 40 cm long are kept at $0°c$ and the initial temperature is $3x + 2$, formulate the model.

Solution:

Boundary conditions:

$$u(0,t) = 0$$
$$u(40,t) = 0$$
$$Initial conditions:$$
$$u(x,0) = 3x + 2$$

47. A rod of length 20 cm whose one end is kept at $30°c$ and the other end at $70°c$, until steady state prevails, find the steady state temperature.

Solution:

Boundary conditions:

$$u(0,t) = 30° --------(1)$$
$$u(20,t) = 70° -------- (2)$$

The steady state solution is

$$u(x) = ax + b ------(3)$$

Applying (1) in (3),

$$u(0) = a.0 + b = 30$$
$$\Rightarrow b = 30$$

Applying (1) in (3),

$$u(20) = a(20) + b = 70° \Rightarrow 20a + 30° = 70°$$
$$\Rightarrow 20a = 40$$
$$\Rightarrow a = 2$$

Substitute "a" and "b" in (3), we get

$$u(x) = 2x + 30$$

48. A bar of length 50 cm has its ends kept at $20°c$ and $100°c$ until steady state prevails. Find the temperature t any point

Solution:

$$u(0,t) = 20°c --------(1)$$
$$u(50,t) = 100°c -------- (2)$$

The steady state solution is

$$u(x) = ax + b ------(3)$$

Applying (1) in (3), we get

$$u(0) = a(0) + b = 20$$
$$\Rightarrow b = 20$$

Applying (2) in (3), we get

$$u(50) = a.50 + b = 100°$$
$$\Rightarrow 50a + 20 = 100$$
$$\Rightarrow 50a = 80$$
$$\Rightarrow a = \frac{8}{5} = 1.6$$
$$\therefore a = 1.6$$

Substitute “a” and “b” in (3), we get

$$u(x) = 1.6x + 20$$

49. Write the general solution of y(x,t) of vibrating motion of length l and fixed end points and zero initial shape.

Solution:

$$y(x,t) = \sum_{1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)\sin\left(\frac{n\pi ct}{l}\right).$$

50. Classify the P.D.E $x^2 u_{xx} + 2xyu_{xy} + (1+y^2)u_{yy} - 2u_x = 0.$

Solution:

$$A = x^2 \,;\, B = 2xy; C = 1 + y^2$$
$$B^2 - 4AC = 4x^2y^2 - 4x^2(1+y^2) = -4x^2 < 0.$$

$\therefore$ The given P.D.E is elliptic.

UNIT V –Z transform

Two marks Q&A

1. Find the Z-transform of a $b^n \ (a, b \neq 0)$ **.**

Solution:

$$Z\{ab^n\} = \sum_{n=0}^{\infty} ab^n z^{-1}$$

$$= a\sum_{n=1}^{\infty}\left(\frac{b}{z}\right)^n = \frac{a}{1-\left(\frac{b}{z}\right)} = \frac{az}{z-b}$$

2. Prove that $Z[f(t+T)] = z[\bar{f}(z) - f(0)]$ **.**

Solution:

$$Z[f(t+T)] = \sum_{n=0}^{\infty} f(nT+T)z^{-n}$$

$$= \sum_{n=0}^{\infty}[f(n+1)T]z^{-n}$$

$$= z\sum_{n=0}^{\infty}[f(n+1)T]z_{-(n+1)}$$

$$= z\sum_{n=0}^{\infty}[f(kT)z^{-k}$$ where k=n+1

$$= z[\bar{f}(z) - f(0)] .$$

3. Find the Z-transform of n?

Solution:

$$Z[n] = \sum_{n=0}^{\infty} n z^{-n} = \sum_{n=0}^{\infty} \frac{n}{z^n}$$

$$= \frac{1}{z} + \frac{2}{z^2} + \frac{3}{z^3} + \ldots\ldots\ldots$$

$$= \frac{1}{z}\left[1 + \frac{2}{z} + \frac{3}{z^2} + \ldots\ldots\right] = \frac{1}{z}\left[1 - \frac{1}{z}\right]^{-2}$$

$$= \frac{z^2}{z(z-1)^2} = \frac{z}{(z-1)^2}.$$

4.state the initial and final value theorem of Z-transform.

Solution:

If $Z[n] = \bar{u}(z)$,then

(i)Initial value theorem: $\lim_{n\to 0} u_n = \lim_{z\to\infty} \bar{u}(z)$

(ii)Final value theorem: $\lim_{n\to\infty} u_n = \lim_{z\to 1} (z-1)\bar{u}(z)$.

5. find $Z\left[\frac{a^n}{n!}\right]$ in z-transform.

Solution:

$$Z\left[\frac{a^n}{n!}\right] = \sum_{n=0}^{\infty} \frac{a^n}{n!} z^{-n} = \sum_{n=0}^{\infty} \frac{(az^{-1})^n}{n!}$$

$$= 1 + \frac{az^{-1}}{1!} + \frac{(az^{-1})^2}{} + \ldots\ldots\ldots$$

$$= e^{az^{-1}} = e^{\frac{a}{z}}$$

6. Find $Z\left[e^{-iat}\right]$ using Z-transform.

Solution:

$$Z\left[e^{-iat}\right] = Z\left[e^{-iat} * 1\right]$$

$$= \left\{Z(1)\right\}_{z \to ze^{iat}} \quad \text{[by shifting property]}$$

$$= \left[\frac{z}{z-1}\right]_{z \to ze^{iat}} \quad \text{[since z(1)=z/z-1]}$$

$$= \frac{ze^{iat}}{ze^{iat} - 1}\ .$$

7. State and prove that initial value thorem in Z-

transform. Solution:

If $\left|Z f(n)\right| = F(z)$ then $\lim_{z\to\infty} F(z) = f(0) = \lim_{t\to 0} f(0)$.

We know that

$$Z\left|f(n)\right| = \sum_{n=0}^{\infty} f(n) z^{-n}$$

$= f(0)+f(1)\, z^{-1} + f(2)\, z^{-2} + \ldots\ldots\ldots$

$$\lim_{z\to\infty} F(z) = \lim_{z\to\infty}[\, f(0) + f(1)/z + f(2)/z^2 + \ldots\ldots\ldots]$$

$= f(0)$

$$\therefore \lim_{z\to\infty} F(z) = \lim_{n\to 0} f(n)$$

8. Find the Z-transform of

(n+1)(n+2). Solution:

$$Z[(n+1)(n+2)] = Z[n^2 + 3n + 2]$$

$$= Z(n^2) + 3Z(n) + 2Z(1)$$

$$= \frac{z(z+1)}{(z-1)^3} + 3\frac{z}{(z-1)^2} + 2\frac{z}{(z-1)}.$$

9. Find the Z-transform of

n? Solution:

$$Z(n) = \sum_{n=0}^{\infty} nz^{-n} = \sum_{n=0}^{\infty} \frac{n}{z^n}$$

$$= \frac{1}{z}\left(1 + \frac{2}{z} + \frac{3}{z^2} + \dots\dots\right)$$

$$= \frac{1}{z}\left(1 - \frac{1}{z}\right)^{-2} = \frac{1}{z}\left(\frac{1}{1-\frac{1}{z}}\right)^2 \quad \left|\frac{1}{z}\right| < 1 \text{ or } |z| > 1$$

$$= \frac{1}{z}\frac{z^2}{(z-1)^2}$$

$$= \frac{z}{(z-1)^2}.$$

10. Find the Z-transform of $\cos n\theta$.

Solution:

$$Z[(e^{i\theta})^n] = \frac{z}{z - e^{i\theta}} = \frac{z}{z - \cos\theta - i\sin\theta} \, x \, \frac{z - \cos\theta + i\sin\theta}{z - \cos\theta + i\sin\theta}$$

$$= \frac{z(z - \cos\theta + i\sin\theta)}{(z - \cos\theta)^2 + \sin^2\theta}$$

Equating real part in both sides

$$Z(\cos n\theta) = \frac{z(z - \cos\theta)}{(z - \cos\theta)^2 + \sin^2\theta}.$$

11. Prove that $Z[(-1)^n] = \frac{z}{z+1}$ **.Also find the region of convergence.**

Solution:

$$Z\{(-1)^n\} = \sum_{n=0}^{\infty} (-1)^n z^{-n} = 1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \ldots\ldots = \left(1 + \frac{1}{z}\right)^{-1}$$

$$= \frac{1}{1 + \frac{1}{z}} = \frac{z}{(z+1)}$$

Here the region of convergence is $\left|\frac{1}{z}\right| 1 \quad or \quad |z| \quad 1$

$$\therefore Z[(-1)^n] = \frac{z}{z+1}.$$

12. Define Z-transform.

Solution:

The z-transform of a sequence {x(n)}is defined by $X(z) = \sum_{n=0}^{\infty} x(n) z^{-n}$,where z is a complex variable.

13. What is the region of convergence? Solution:

The region in which the series $\sum_{n=0}^{\infty} x(n) z^{-n}$ is convergent is called the region of convergence.

14. Find $Z[u(n-1)]$

Solution:

$$Z[u(n-1)] = z^{-1} Z(u(n))$$

$$= \frac{1}{z}\frac{z}{(z-1)} = \frac{1}{(z-1)} \qquad if\,|z \quad 1|$$

15. Find $Z[3^n \delta(n-2)]$ **.**

Solution:

$$Z[3^n \delta(n-2)] = Z[\delta(n-2)]_{z \to \frac{z}{3}}$$

$$= [z^{-2}]_{z \to \frac{z}{3}} = \left(\frac{1}{z^2}\right)_{z \to \frac{z}{3}}$$

$$= \frac{9}{z^2}$$

16. Find $Z^{-1}\left[\frac{2}{2z-1}\right]$

Solution:

$$Z^{-1}\left[\frac{2}{2z-1}\right] = Z^{-1}\left[\frac{1}{\frac{1}{2}}\right] = = \left[\left(\frac{1}{2}\right)^n\right]_{n \to n-1} = \left(\frac{1}{2}\right)^{n-1}$$

$$= Z^{-1}\left[\left(\frac{1}{z}\right)\frac{1}{z-\frac{1}{2}}\right]$$

$$= Z^{-1}\left[z^{-1}\left(\frac{z}{z-\frac{1}{2}}\right)\right]$$

$$= \left[\left(\frac{1}{2}\right)^{n}\right]_{n\to n-1} \qquad = \left(\frac{1}{2}\right)^{n-1}$$

17.Find $Z^{-1}\left[\frac{1}{z+1}\right]$

Solution:

$$Z^{-1}\left[\frac{1}{z+1}\right] = Z^{-1}\left[\frac{1}{z}\left(\frac{z}{z+1}\right)\right]$$

$$= Z^{-1}\left[z^{-1}\left(\frac{z}{z+1}\right)\right]$$

$$= Z^{-1}\left[\left(\frac{z}{z+1}\right)\right]_{n\to n-1}.$$

$$= (-1)^{n-1},\ n = 1,2,3,....$$

18. Find $Z[a^{n}\cos n\pi]$

Solution:

$$Z[a^{n}\cos n\pi] = Z[a^{n}(-1)^{n}]$$

$$= Z[(-1)^{n}]_{z\to \frac{z}{a}}$$

$$= \left(\frac{z}{z+1}\right)_{z \to \frac{z}{a}} = \frac{z}{z+a}$$

19.Find $Z[e^{at+b}]$

Solution:

$$Z[e^{at+b}] = e^{b} Z[e^{at}]$$

$$= e^{b} \frac{z}{z - e^{at}}$$

20.Find the Z-transform of the convolution of x(n)= $a^{n} u(n) and y(n) = b^{n} u(n)$

Solution:

Z[x(n)*y(n)=Z[x(n)]Z[y(n)]= $\frac{z}{z-a}\frac{z}{z-b} = \frac{z^{2}}{(z-a)(z-b)}$

21.State second shifting theorem in Z-transform.

Solution:

If Z[f(n)]=F(z),then

Z[f(n+k)]= $z^{k}\left[F(z) - f(0) - \frac{f(1)}{z} - \frac{f(2)}{z^{2}} - \ldots - \frac{f(k-1)}{z^{k-1}}\right]$

22.Find $Z^{-1}\left[\frac{z}{z^2+4z+4}\right]$

Solution:

$$Z^{-1}\left[\frac{z}{z^2+4z+4}\right] = .\ Z^{-1}\left[\frac{z}{(z+2)^2}\right].$$

$$=\left(\frac{-1}{2}\right) Z^{-1}\left[\frac{(-2)z}{(z+2)^2}\right]=\left(\frac{-1}{2}\right)n(-2)^n = n(-2)^{n-1}$$

$$=\left(\frac{-1}{2}\right)n(-2)^n = n(-2)^{n-1}$$

23.Find $Z^{-1}\left[\frac{z}{(z-1)(z-2)}\right]$

Solution:

$$Z^{-1}\left[\frac{z}{(z-1)(z-2)}\right]= Z^{-1}\left[\frac{z}{z-2}-\frac{z}{z-1}\right] = 2^n - 1$$

24. Find the z-transform of $\left(\frac{1}{3}\right)^n u(n)$

Solution:

Z$\left[\left(\frac{1}{3}\right)^n u(n)\right]$=Z$\left[3^{-n}u(n)\right]$

$$=\left(\frac{z}{z-1}\right)_{z\to 3z}$$

$$=\frac{3z}{3z-1}$$

$$= \frac{z}{z - \frac{1}{3}}$$

25. . Find the z-transform of $n2^n$

Solution:

Z[$n2^n$]= $Z(n)_{z \to \frac{z}{2}}$

$$= \left[\frac{z}{(z-1)^2}\right]_{z \to \frac{z}{2}} = \frac{2z}{(z-2)^2}$$

26.State the damping rule in Z-transform

Solution:

If Z[x(n)]=X(z),then

(i) $Z[a^{-n} x(n)] = X(az) and$

$(ii) Z[a^n x(n)] = X(\frac{z}{a})$

27.If $Z[y_n] = Y(z)$ **,then write down the values of of** $Z[y_{n-k}] and Z[y_{n+k}]$

Solution:

$Z[y_{n-k}] = z^{-k} Y(z)$ and

$$Z[y_{n+k}] = z^k \left[Y(z) - y(0) - \frac{y(1)}{z} - \frac{y(2)}{z^2} - ......- \frac{y(k-1)}{z^{k-1}}\right]$$

28. If Z[x(n)]=X(z),then what are the values of Z[x(n+1)] and Z[x(n+2)] Solution:

Z[x(n+1)]=z{X(z)-x(0)] and

Z[x(n+2)]= $z^2 [X(z) - x(0) - \frac{x(1)}{z}]$

29.State convolution theorem for Z-transform

Solution:

If $Z^{-1}[X(z)] = x(n) and Z^{-1}[Y(z)] then$

$Z^{-1}[X(z)Y(z)] = x(n)^* y(n) = \sum^{n}_{k=0} x(k)\, y(n-k)$

30.Find $Z^{-1}[\frac{z}{(z-3)(z-4)}]$

Solution:

$$Z^{-1}[\frac{z}{(z-3)(z-4)}] = Z^{-1}[\frac{z}{(z-4)} - \frac{z}{z-3}]$$

$$= Z^{-1}[\frac{z}{(z-4)}] - Z^{-1}[\frac{z}{(z-3)}]$$

$$= 4^n - 3^n$$

31.Solve $y_{n+1} - 2y_n = 1 given y_0 = 0$,
Solution:

Taking Z-transform on both sides

$$(z - 2)Y(z) = \frac{z}{z-1} \Rightarrow Y(z) = [\frac{z}{(z-1)(z-2)}]$$

$$\Rightarrow Y(z) = \frac{z}{z-2} - \frac{z}{z-1}$$

$$y_n = Z^{-1}[Y(z)] = 2^n - 1$$

32. Solve $y_{n+1} - 2y_n = 0 \; given \; y_0 = 2$

Solution:

Taking Z-transform

(z-2)z(y(n))=2z

Z[y(n)]=2 $\frac{z}{z-2}$

Y(n)= $z^{-1}[\frac{2z}{z-2}] = 2.2^n = 2^{n+1}$

33.Find $3^n * 3^n$ **using Z-transform**

Solution:

Z[$3^n * 3^n$]= $Z(3^n).Z(3^n) = \frac{z^2}{(z-3)^2}$

$3^n * 3^n = Z^{-1}[\frac{z^2}{(z-3)^2}] = Z^{-1}[\frac{z(z-3+3)}{(z-3)^2}]$

$= Z^{-1}[\frac{z}{z-3}] + Z^{-1}[\frac{3z}{(z-3)^2}]$

$= 3^n + n3^n = 3^n(n+1)$

34.Find the Z-transform of $2^n * n$

Solution:

$$Z[2^n * n] = Z[(2^n) Z(n) = \frac{z}{z-2} \cdot \frac{z}{(z-1)^2} = \frac{z^2}{(z-2)(z-1)^2}]$$

35. Find the Z-transform of u(n-2)

Solution:

$$Z[u(n-2)] = z^{-2} Z[u(n)] = \frac{1}{z^2} \cdot \frac{z}{z-1} = \frac{1}{z(z-1)}$$

36. Find the Z-transform of $2^n u(n-1)$

Solution:

$$Z[2^n u(n-1)] = Z[[u(n-1)]_{z \to \frac{z}{2}}]$$

$$Z[u(n-1) = \frac{1}{z} Z[u(n)] = \frac{1}{z-1}$$

$$Z[[2^n u(n-1)] = \frac{1}{(\frac{z}{2}-1)} = \frac{2}{z-2}$$